Teoría de Expansiones Asintóticas Acopladas

Imaginemos que tenemos que resolver el siguiente problema adimensionalizado :

$$\epsilon u'' + u u' + u^2 = 0 \qquad 0 \leq x \leq 1 \; ; \; \epsilon \ll 1$$

$$x = 0 : \; u = 0$$

$$x = 1 : \; u = 1$$

Solución de orden 0 o de Euler $(\epsilon = 0)$:

$$u u' + u^2 = 0 \quad \text{ORDEN } 1 \longrightarrow \text{SÓLO 1 C.C. EN } x$$

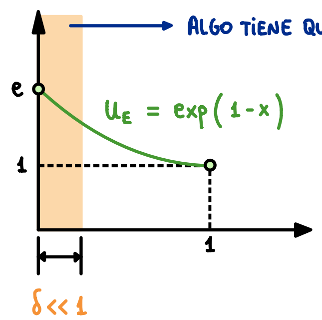
$$\begin{cases} u_E = 0 \; \times \\ u_E' + u_E = 0 \; \checkmark \longrightarrow u_E = \boxed{C_1} \exp(-x) \end{cases}$$

Si imponemos $x = 0 : u = 0 \longrightarrow x = 0 : u_E = 0 \longrightarrow C_1 = 0 \; \times$

Si imponemos $x = 1 : u = 1 \longrightarrow x = 1 : u_E = 1 \longrightarrow C_1 = e \; \checkmark \longrightarrow \boxed{u_E = \exp(1-x)}$

DESTRUYE LA HOMOGENEIDAD

$$\left.\begin{array}{l} u_E(x=0) = e \\ u_E(x=1) = 1 \end{array}\right\} \longrightarrow \frac{u_E(x=0) - u_E(x=1)}{u_E(x=0)} \sim O(1) \longrightarrow$$ PROBLEMA DE PERTURBACIONES SINGULARES (PPS)



$\longrightarrow$ ALGO TIENE QUE PASAR AQUÍ PARA CUMPLIR LA C.C. EN $x = 0 \longrightarrow$ ACTIVAR $\epsilon u'' \longrightarrow$

$\longrightarrow$ NUEVA ESCALA δ $\left(\frac{\delta}{\ell} \ll 1\right)$ CERCA DE $x = 0$.

Definimos $\xi = \dfrac{x}{\delta(\epsilon)}$ de manera que cuando las variaciones de x sean de orden δ, las variaciones de ξ sean de orden unidad $\Delta x \sim \delta \longrightarrow \Delta \xi \sim 1$ (ZOOM).

Multiplicando por δ :

$$\underbrace{\frac{\epsilon}{\delta}}_{\boxed{\sim\frac{\epsilon}{\delta}}}\overbrace{\underbrace{\frac{d^2u}{d\xi^2}}}^{\sim 1} + \underbrace{u\overbrace{\frac{du}{d\xi}}^{\sim 1}}_{\sim 1} + \underbrace{\delta u^2}_{\sim\delta} = 0 \longrightarrow \text{3 CASOS}$$

- $\frac{\epsilon}{\delta} \ll 1 \longrightarrow$ volvemos a Euler ✗
- $\frac{\epsilon}{\delta} \sim 1 \longrightarrow \underbrace{\frac{\epsilon}{\delta}\frac{d^2u}{d\xi^2}}_{\text{ACTIVADO}} + \underbrace{u\frac{du}{d\xi}}_{\text{PERMITE EMPALMAR CON EULER}} = 0$ ✓ LÍMITE DISTINGUIDO DE δ
- $\frac{\epsilon}{\delta} \gg 1 \longrightarrow$ término descompensado, no podemos conectar con Euler ✗

Por tanto :

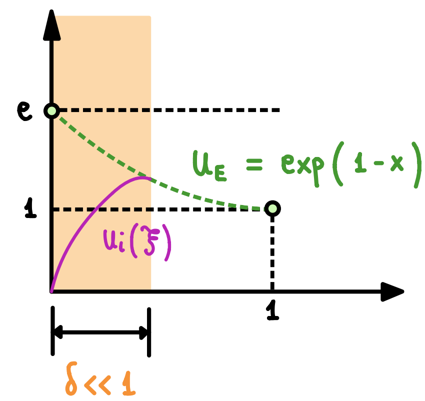
$$\delta(\epsilon)\sim\epsilon \xrightarrow{\text{Para simplificar}} \boxed{\delta(\epsilon) = \epsilon}$$

$$\underbrace{\frac{d^2u}{d\xi^2}}_{\sim 1} + \underbrace{u\frac{du}{d\xi}}_{\sim 1} + \underbrace{\cancel{\epsilon u^2}}_{\sim\epsilon\ll 1} = 0 \xrightarrow{u|_{x\sim\delta\sim\epsilon} = u_i}$$

$$\longrightarrow \boxed{\begin{gathered}\frac{d^2u_i}{d\xi^2} + u_i\frac{du_i}{d\xi} = 0\\ \xi = 0:\ u_i = 0\\ \xi \gg 1:\ u_i \to u_E|_{x\to 0} = e\end{gathered}}$$

EMPALME CON EULER CERCA DE $x = 0$:

$$u_i\Big|_{\substack{\xi\to\infty\\(\xi\gg 1)}} = u_E\Big|_{\substack{x\to 0\\(x\ll 1)}}$$



Truco : $\frac{du_i}{d\xi} = q \longrightarrow \frac{d^2u_i}{d\xi^2} = \frac{dq}{d\xi} = \frac{dq}{du_i}\underbrace{\frac{du_i}{d\xi}}_{\xi} = \xi\frac{dq}{du_i} \longrightarrow \xi\frac{dq}{du_i} + u_i q = 0 \longrightarrow$

$$\longrightarrow q\left(\frac{dq}{du_i} + u_i\right) = 0 \quad \begin{cases} q = 0 \xrightarrow{\xi = 0:\ u = 0} u_i = 0 \ ✗ \\ \frac{dq}{du_i} = -u_i \longrightarrow q = a - \frac{u_i^2}{2} = a\left[1 - \left(\frac{u_i}{\sqrt{2a}}\right)^2\right] = \frac{du_i}{d\xi} \longrightarrow \end{cases}$$

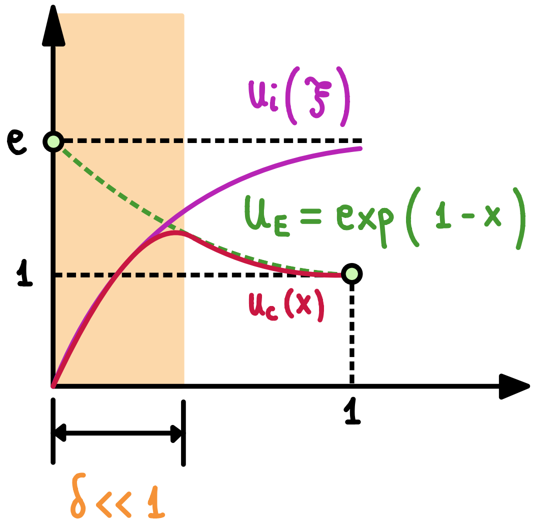
$$\longrightarrow \int \frac{d\left(\frac{u_i}{\sqrt{2a}}\right)}{1-\left(\frac{u_i}{\sqrt{2a}}\right)^2} = \int\sqrt{\frac{a}{2}}\,d\xi \xrightarrow{\frac{1}{1-■^2} = \frac{1}{2}\left(\frac{1}{1-■} + \frac{1}{1+■}\right)} \underbrace{\frac{1}{2}\int\frac{\varphi}{1-\varphi}d\varphi} + \underbrace{\frac{1}{2}\int\frac{\varphi}{1+\varphi}d\varphi} \longrightarrow$$

$\xi \gg 1:\ u_i \to u_E|_{x\to 0} = e \quad \begin{cases}\varphi \to \frac{e}{\sqrt{2a}} \\ \exp(\sqrt{2a}\,\xi) \to \cancel{\infty}\quad \varphi = 1 \text{ LO BLOQUEA}\end{cases} \longrightarrow \boxed{a = \frac{e^2}{2}}$

Entonces:

$$\frac{1+\frac{u_i}{e}}{1-\frac{u_i}{e}} = \exp(e\xi) \longrightarrow \boxed{\frac{e+u_i}{e-u_i} = \exp(e\xi)}$$

Solución compuesta: $u_c(x) = u_E(x) + u_i(\xi) - \underbrace{u_E\Big|_{x\to 0}}_{u_e = e} \longrightarrow \boxed{u_c(x) = u_E(x) + u_i\left(\frac{x}{\epsilon}\right) - e}$



Comprobación de validez:

$x \sim 1: \; u_c(x) = u_E(x) + \underbrace{\cancel{u_i(\xi >> 1)}}_{u_e} - \cancel{u_e} = u_E(x)$ ✓

$x >> 1: \; u_c(x) = \underbrace{\cancel{u_E(x << 1)}}_{u_e} + u_i\left(\frac{x}{\epsilon}\right) - \cancel{u_e} = u_i\left(\frac{x}{\epsilon}\right)$ ✓